A word on uniqueness

Since there are different ways you can put A into U.T. form (using different row operations), you might wonder: is it possible to get different values of $\det A$ if you do two different row reductions?

The answer is no. The determinant is unique.

Why?

We've defined the determinant to be a function $\det: \{n \times n \text{ matrices}\} \to \mathbb{R}$ (multilinear function / linear in each row) that satisfies 3 axioms

It turns out that there is a unique function satisfying these axioms and so det is unique.

To show this, you first come up with a general formula for $\det A$ and then show that any function satisfying the axioms can be computed with this formula.

We will think about such a formula soon, but we won't explicitly prove this fact

(come to office hours if you'd like to know more)

Theorem: A is invertible $\iff \det A \neq 0$

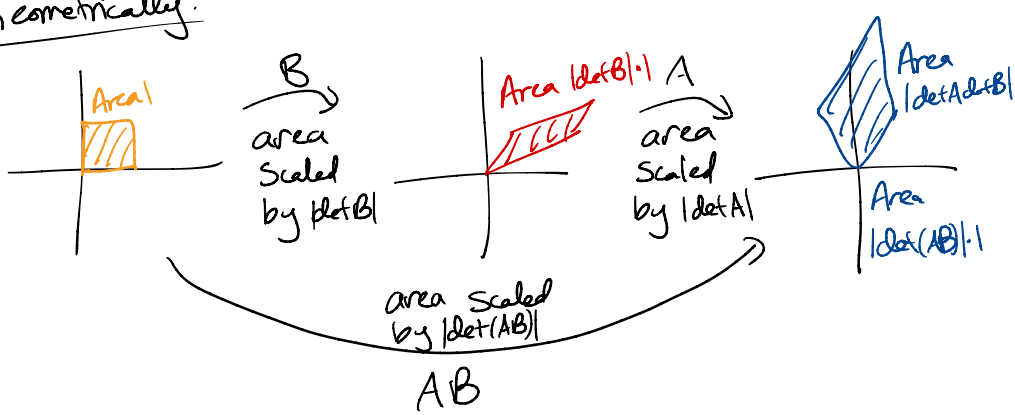
Proof: Reduce A to U.T. form U using elimination & row swaps. Then:

$$|\det A| = |u_{11} \cdots u_{nn}| = \begin{cases} 0 & \text{if } A \text{ doesn't have } n \text{ pivots} \\ \text{nonzero} & \text{if } A \text{ has } n \text{ pivots} \end{cases}$$

Hence $\det A \neq 0 \iff A$ has n pivots
$\iff A$ is invertible. ∎

Property: $\det(AB) = (\det A)(\det B)$

Proof: Geometrically:



Axiomatically:

Consider the function $D(B) = \frac{\det(AB)}{\det A}$.

This satisfies the three axioms (check)

Hence, $D(B) = \det(B) \implies \det B = \frac{\det(AB)}{\det A}$

$\implies \det AB = \det A \cdot \det B$ ∎

More Properties

a) $\det(A^{-1}) = \frac{1}{\det A}$

b) $\det(A^T) = \det A$

Proof:

a) $1 = \det I = \det(AA^{-1}) = \det A \cdot \det A^{-1} \Rightarrow \det A = \frac{1}{\det A^{-1}}$

b) Consider a $PA = LU$ factorization.

Then $(PA)^T = (LU)^T$ or $A^T P^T = U^T L^T$

Note: $\det L = 1$ and $\det L^T = 1$ (since 1's on diagonal)

$\det(U) = \det(UI^T)$ (since both triangular w/ same diagonal)

$\det(P) = (-1)^r = \det(P^T)$ (HW)

So $\det(PA) = \det(LU)$

$\Rightarrow \det P \det A = \det L \det U$

$\Rightarrow \det A = \frac{\det L \det U}{\det P} = \frac{\det L^T \det U^T}{\det P^T}$

Similarly, $\det(A^T P^T) = \det(U^T)\det(L^T)$

$\Rightarrow \det(A^T) = \frac{\det(U^T)\det(L^T)}{\det(P^T)}$

Hence $\det(A^T) = \det A$ ∎

Determinant Formulas

Let's use the axioms to derive $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \det\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$

Axiom3

$$= a\det\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} + b\det\begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$$

Axiom3

Axiom3 →

$$= a\left(c\det\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + b\left(c\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\det\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

$\det\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 0$ by Prop 2

$\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ by Axiom 1

$\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$ by Axiom 2

$\det\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0$ by Prop 2

$$= ad - bc$$

We can do this to derive a formula the determinant for any square matrix

Cofactor expansions repackage this into a recursive formula

Def: Let A be an $n \times n$ matrix,

Let A_{ij} be the $(n-1)\times(n-1)$ matrix obtained by removing the ith row and jth column of A

Let $C_{ij} = (-1)^{i+j} \det A_{ij}$. this is a cofactor of A

Recursive Formula: $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

cofactor expansion along the first row

Ex: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $C_{11} = (-1)^{1+1} \det[d] = d$

$C_{12} = (-1)^{1+2} \det[c] = -c$

So $\det A = aC_{11} + bC_{12} = ad - bc$ ✓